THE ENTROPY CONJECTURE FOR PARTIALLY HYPERBOLIC DIFFEOMORPHISMS WITH 1-D CENTER

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ABSTRACT. We prove that if f is a partially hyperbolic diffeomorphism on the compact manifold M with one dimensional center bundle, then the logarithm of the spectral radius of the map induced by f on the real homology groups of M is smaller or equal to the topological entropy of f. This is a particular case of the Shub's entropy conjecture, which claims that the same conclusion should be true for any C^1 map on any compact manifold.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let M be a m-dimensional compact Riemannian manifold without boundary and let $f\colon M \to M$ be a differentiable map.

The map f will induce a linear action on the real homology groups of M, denoted $f_{*,k}\colon H_k(M,\mathbb{R}) \to H_k(M,\mathbb{R})$. The *spectral radius* of these maps are denoted $sp(f_{*,k})$ and they are equal to the largest eigenvalue in absolute value of the linear map $f_{*,k}$. The *spectral radius of* f_* is

$$sp(f_*) = \max_k sp(f_{*,k}).$$

We will also use the common notation $h(f)$ for the *topological entropy* of f, for a definition we send the reader to [HK] for example.

The diffeomorphism f is called *partially hyperbolic* if there exist an invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$, with at least two subbundles nontrivial, and there exist $\alpha, \beta > 1$, $C, D > 0$ such that:

(1) E^u is uniformly expanding:

$$\|Df^k(v_u)\| \geq C\alpha^k \|v_u\|, \quad \forall\, v_u \in E^u, k \in \mathbb{N},$$

(2) E^s is uniformly contracting:

$$\|Df^k(v_s)\| \leq D\beta^{-k} \|v_s\|, \quad \forall\, v_s \in E^s, k \in \mathbb{N},$$

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(3) E^u dominates E^c, and E^c dominates E^s:

$$\|Df|_{E^s_x}\| < \|Df|^{-1}_{E^c_x}\|^{-1} \leq \|Df|_{E^c_x}\| < \|Df|^{-1}_{E^u_x}\|^{-1}, \quad \forall\, x \in M.$$

Condition (3) could be replaced with some weaker condition, of eventual domination for a power of f, but this doesn't make any difference in the following considerations, because by taking that power of f or by changing the Riemannian metric on M we can always assume this strong domination condition.

We will prove the following result:

Theorem 1. *Suppose that M is a compact Riemannian manifold without boundary and $f\colon M \to M$ is a partially hyperbolic diffeomorphism with one dimensional center bundle. Then*

$$h(f) \geq \log sp(f_*).$$

We will prove the theorem in the next section. We remark that this is a special case of the entropy conjecture formulated by Shub in [Sh]:

Conjecture 1. *In f is a C^1 map on the compact manifold without boundary M then*

$$h(f) \geq \log sp(f_*).$$

This conjecture was proven for C^∞ maps by Yomdin ([Yo]), and it is not true for Lipschitz maps ([Pu]). It is also true if M is an infra-nilmanifold for C^0 maps (Marzantowicz and Przytycki, [MP2]), or a manifold of dimension at most three for C^1 maps (combine [MP] with [Ma] and use duality). There are other weaker versions known to be true, when one replaces the spectral radius of f_* by some smaller invariants: the degree for C^1 maps (Misiurewicz and Przytycki, [MP]), the spectral radius on the first homology group for C^0 maps (Manning, [Ma]), the growth on the fundamental group for C^0 maps (Bowen, [Bo]), the asymptotic Nielsen number for C^0 maps (Ivanov, [Iv]).

The conjecture is also true for diffeomorphisms satisfying the Axiom A and no-cycle conditions, so in particular it is true for Anosov diffeomorphisms (Shub and Williams, [SW]; Ruelle and Sullivan, [RS]). The partially hyperbolic diffeomorphisms are natural generalizations of hyperbolic diffeomorphisms, and it is expected that they have similar properties, at least in the generic setting and/or for small dimensions of the center distribution. Our result is another fact that support this claim.

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2. Proofs

In this section we will prove the Theorem 1. We will use two propositions interesting on their own right which we will state after we introduce some notions.

Suppose $TM = E \oplus F$ is a *dominated splitting* for f, in the sense that

$$m(Df|_{F_x}) := \|Df|_{F_x}^{-1}\|^{-1} < \|Df|_{E_x}\|, \quad \forall x \in M.$$

Denote by $\mathcal{T}(E)$ the family of C^1 disks in M uniformly transverse to E (the angle between the tangent plane to the disk and E is bounded away from zero) and with the same dimension as F:

$$\mathcal{T}(E) = \{D \subset M,\, C^1 \text{ disk } : \dim D = \dim F, D \pitchfork E, \inf_{x \in D} \angle(T_x D, E_x) > 0\}.$$

Define the *volume growth of a disk D under f* to be the exponential rate of growth of the volume of the iterates of the disk:

$$\chi(D, f) = \limsup_{n \to \infty} \frac{\log(\text{ vol } (f^n(D)))}{n},$$

and *the volume growth of $\mathcal{T}(E)$ under f*:

$$\chi(\mathcal{T}(E), f) = \sup_{D \in \mathcal{T}(E)} \chi(D, f).$$

The first proposition relates the volume growth of $\mathcal{T}(E)$ under f with the topological entropy of f:

Proposition 2. *Suppose $TM = E \oplus F$ is a dominated splitting for f. Then the topological entropy of f is greater or equal to the volume growth of $\mathcal{T}(E)$:*

$$h(f) \geq \chi(\mathcal{T}(E), f).$$

Proof. We have to prove that for every disk $D \in \mathcal{T}(E)$ we have $h(f) \geq \chi(D, f)$. Because $\chi(A \cup B, f) = \max\{\chi(A, f), \chi(B, f)\}$, we may assume that the disk D is arbitrarily small in diameter. Because $\chi(D, f) = \chi(f^n(D), f)$ and

$$\lim_{n \to \infty} \angle(T_{f^n(x)} f^n(D), F_{f^n(x)}) = 0$$

uniformly with respect to $x \in D$ (this is because the splitting is dominated and the starting disk D is transversal to E), we may also assume that $\angle(T_y f^n(D), F_y) < \frac{\epsilon}{2}$ for all $n \geq 0$ and $y \in f^n(D)$, and some fixed $\epsilon > 0$ small. A dominated splitting is also continuous, so we can assume that there is $\delta > 0$ such that if $x, y \in f^n(D)$ with $d(x, y) < \delta$ then $\angle(T_y f^n(D), F_x) < \epsilon$. Here d is the Riemannian metric on the manifold M. This implies that at the scale δ the Riemannian metric d on M is equivalent to the Riemanian metric $\tilde{d}$ induced on the submanifolds $f^n(D)$, meaning that there exists $C > 0$ such that if $x, y \in f^n(D)$ for some n and $\tilde{d}(x, y) < \delta$ then

$$d(x, y) \leq \tilde{d}(x, y) \leq Cd(x, y).$$

This can be proved using some small charts and eventually making δ slightly smaller. In the same way one can prove that for any $\delta' < \delta$ there is an upper bound $B_{\delta'} > 0$ for the volumes of the balls in $f^n(D)$ of $\tilde{d}$-radius δ', independent of n:

$$\text{vol }(B_{\tilde{d}}(x, \delta')) \leq B_{\delta'}, \quad \forall\, x \in f^n(D), n \geq 0.$$

Now let $K = \sup_{x\in M} \|Df_x\|$ and choose $\delta' > 0$ such that $C\delta' < \frac{\delta}{K}$, and assume that $diam_{\tilde{d}}(D) < C\delta'$. Let S_n be a maximal $C\delta'$-separated set in $f^n(D)$ w.r.t. $\tilde{d}$. Then

$$f^n(D) \subset \bigcup_{x\in S_n} B_{\tilde{d}}(x, C\delta'),$$

so

$$\text{vol }(f^n(D)) \leq \sum_{x\in S_n} \text{vol }(B_{\tilde{d}}(x, C\delta')) \leq B_{C\delta'}|S_n|,$$

where $|S_n|$ is the cardinality of S_n. Now suppose that $x, y \in f^{-n}S_n$, so $\tilde{d}(x, y) < C\delta'$ and $\tilde{d}(f^n(x)f^n(y)) > C\delta'$. Then there exist $k \in \{0, 1, 2, \ldots n-1\}$ such that:

$$\tilde{d}(f^k(x)f^k(y)) \leq C\delta' \quad \tilde{d}(f^{k+1}(x)f^{k+1}(y)) > C\delta'.$$

Then

$$\tilde{d}(f^{k+1}(x)f^{k+1}(y)) \leq K\tilde{d}(f^k(x)f^k(y)) < \delta$$

so

$$d(f^{k+1}(x)f^{k+1}(y)) \geq \frac{1}{C}\tilde{d}(f^{k+1}(x)f^{k+1}(y)) > \delta',$$

which means that the set $f^{-n}(S_n)$ is (n, δ')-separated w.r.t. d. So if we denote by $N(n, \delta', f)$ the maximal cardinality of a (n, δ')-separated set for f, we get that

$$N(n, \delta', f) \geq |S_n| \geq \frac{1}{B_{C\delta'}} \text{ vol }(f^n(D)).$$

But this implies that $h(f) \geq \chi(D, f)$ and consequently

$$h(f) \geq \chi(\mathcal{T}(E), f). \qquad \square$$

The second proposition relates the volume growth of $\mathcal{T}(E)$ under f with the spectral radii of $f_{*,l}$ for $l \leq \dim F$ in the case when F is uniformly expanding:

Proposition 3. *Suppose that $TM = E \oplus F$ is a dominated splitting for f and F is uniformly expanding under Df. Then for any $l < \dim F$ we have:*

$$\log(sp(f_{*,l})) < \chi(\mathcal{T}(E), f),$$

and for $\dim F$ *we have:*

$$\log(sp(f_{*,\dim F})) \leq \chi(\mathcal{T}(E), f).$$

Proof. Let $\dim F = u$. First we will prove that $\log(sp(f_{*,u})) \leq \chi(\mathcal{T}(E), f)$.

Let $\sigma = \sum_{i=1}^{p} a_i \sigma_i$, $a_i \in \mathbb{R}$, be a u-dimensional cycle corresponding to an eigenvalue of $f_{*,u}$ with maximal absolute value. Let ω be a dual differential form, so

$$\limsup_{n\to\infty} |f_*^n \sigma(\omega)|^{\frac{1}{n}} = sp(f_{*,u}).$$

This is true if the eigenvalue is both real or complex. We can also assume that σ is transverse to E, meaning that each disk (simplex) σ_i is transverse to E. Now

$$\begin{aligned}
\log(sp(f_{*,u})) &= \limsup_{n\to\infty} \frac{1}{n} \log |\, f_*^n \sigma(\omega)\,| \\
&= \limsup_{n\to\infty} \frac{1}{n} \log \Big| \sum_{i=1}^{p} a_i \int_{f^n(\sigma_i)} \omega \Big| \\
&\leq \limsup_{n\to\infty} \frac{1}{n} \log \Big(\sum_{i=1}^{p} \text{ vol } (f^n(\sigma_i)) \Big) \\
&= \max_{1\leq i\leq p} \chi(\sigma_i, f) \\
&\leq \chi(\mathcal{T}(E), f).
\end{aligned}$$

Here we used the fact that $\left|\int_D \omega\right| \leq C \text{ vol } (D)$ and the constants disappear in the limit after taking the log and dividing by n. We should remark here that for this inequality which we obtained in the case $l = \dim F$ we didn't use neither the dominated splitting nor the uniform expansion of F.

Now assume that $l < u$ and we will prove that $\log(sp(f_{*,l})) < \chi(\mathcal{T}(E), f)$.

Let $\sigma = \sum_{i=1}^{p} a_i \sigma_i$, $a_i \in \mathbb{R}$, be again a l-dimensional cycle corresponding to an eigenvalue of $f_{*,l}$ with maximal absolute value, and η be a dual differential form, so

$$\limsup_{n\to\infty} |f_*^n \sigma(\eta)|^{\frac{1}{n}} = sp(f_{*,l}).$$

Again we can assume that $\sigma_i \pitchfork E$.

Let $K = \cup_{i=1}^{p} \sigma_i$ be the geometric complex corresponding to σ, with the Riemannian metric as submanifolds of M on each σ_i and the corresponding measure m_i. Let $D = [0,1]^{u-l}$ be the unit cube in $\mathbb{R}^{u-l}$ with the

Lebesgue measure m_D. Following [SW], one can construct a continuous map $H\colon K \times D \to M$ such that:

(1) $H(\cdot, 0) = id_K$;
(2) $H|_{\sigma_i \times D}$ is a diffeomorphism from $\sigma_i \times D$ to $D_i := H(\sigma_i \times D) \subset M$;
(3) D_i is transverse to E, or $D_i \in \mathcal{T}(E)$.

For each $y \in D$ consider the cycle in M

$$\sigma_y = \sum_{i=1}^{p} a_i H(\sigma_i \times \{y\}).$$

Because for every $y \in D$ the cycles σ_y and σ are homotopic, they will have the same homology, so we have:

$$\sigma_y(f^{*n}\eta) = \sigma(f^{*n}\eta).$$

Then

$$\begin{aligned}
\log(sp(f_{*,l})) &= \limsup_{n\to\infty} \frac{1}{n} \log |f_*^n \sigma(\eta)| = \limsup_{n\to\infty} \frac{1}{n} \log |\sigma(f^{*n}\eta)| \\
&= \limsup_{n\to\infty} \frac{1}{n} \log \int_D |\sigma_y(f^{*n}\eta)| dm_D \\
&= \limsup_{n\to\infty} \frac{1}{n} \log \int_D \Big| \sum_{i=1}^{p} a_i \int_{H(\sigma_i \times \{y\})} f^{*n}\eta \Big| dm_D \\
&= \limsup_{n\to\infty} \frac{1}{n} \log \int_D \Big| \sum_{i=1}^{p} a_i \int_{\sigma_i \times \{y\}} H^* f^{*n}\eta \Big| dm_D \\
&\le \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^{p} |a_i| \int_D \Big| \int_{\sigma_i \times \{y\}} H^* f^{*n}\eta \Big| dm_D \\
&\le \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^{p} |a_i| \int_D \int_{\sigma_i \times \{y\}} \| H^* f^{*n}\eta|_{T(\sigma_i \times D)} \| dm_i dm_D \\
&= \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^{p} |a_i| \int_{\sigma_i \times D} \| H^* f^{*n}\eta|_{T(\sigma_i \times D)} \| d(m_i \times m_D).
\end{aligned}$$

But now we know that H is a diffeomorphism from $\sigma_i \times D$ to D_i, so the Jacobian is uniformly bounded away from zero and infinity, and H^* also affects the norm of differential forms in an uniformly bounded way. Denote by m_{D_i} the Riemannian measure on D_i. Because again the constants will disappear in the limit we get:

$$\log(sp(f_{*,l})) \leq \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^{p} \int_{D_i} \|f^{*n}\eta|_{TD_i}\| dm_{D_i}.$$

Because F is uniformly expanding there exist $\lambda > 1$ and $C > 0$ such that:

$$\|Df^n(v)\| \geq C\lambda^n \|v\| , \quad \forall\, v \in F.$$

Because $TM = E \oplus F$ is a dominated splitting then the same is true for all the vectors inside some small invariant cone field around F. By taking iterates if necessary, we may also assume that the disks D_i are tangent to this cone field, so the same relation holds for vectors in TD_i. But this in turn implies that the ratio between the u-dimensional volume expansion on TD_i, or the Jacobian of f restricted to D_i - $|Df|_{TD_i}|$, and the maximal l-dimensional volume expansion on TD_i under n iterates of f is greater than $C^{u-l}\lambda^{n(u-l)}$, and consequently

$$\|f^{*n}\eta|_{TD_i}\| \leq \frac{C'}{\lambda^{n(u-l)}} \left|Df|_{TD_i}\right|.$$

So going back to the logarithm of the spectral radius, we get:

$$\begin{aligned}
\log(sp(f_{*,l})) &\leq \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^{p} \int_{D_i} \frac{C'}{\lambda^{n(u-l)}} \left|Df|_{TD_i}\right| dm_{D_i} \\
&= -(u-l)\log\lambda + \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^{p} \int_{D_i} \left|Df|_{TD_i}\right| dm_{D_i} \\
&= -(u-l)\log\lambda + \limsup_{n\to\infty} \frac{1}{n} \log \sum_{i=1}^{p} \ \mathrm{vol}\ (f^n(D_i)) \\
&= -(u-l)\log\lambda + \max_{1\leq i\leq p} \chi(D_i, f) < \chi(\mathcal{T}(E), f). \qquad \square
\end{aligned}$$

Now we can give the proof of the theorem.

Proof of Theorem 1. First we make the observation that it is enough to prove the result for finite covers of M, so by taking a double cover if necessary, we can assume that M is orientable (see [SW]).

Denote $m := \dim(M)$, $u := \dim(E^u)$ and $s := \dim(E^s)$. Because the center bundle is one-dimensional we have

$$m = u + s + 1.$$

Then $TM = E^{cs} \oplus E^u$, where $E^{cs} = E^s \oplus E^c$, is a dominated splitting for f, so by Proposition 2 we have

$$\chi(\mathcal{T}(E^{cs}), f) \leq h(f).$$

E^u is also uniformly expanding, so by Proposition 3 we have

$$\log(sp(f_{*,l})) \leq \chi(\mathcal{T}(E^{cs}), f), \quad \forall\, 0 \leq l \leq u.$$

Putting these two inequalities together we get

$$\log(sp(f_{*,l})) \leq h(f), \quad \forall\, 0 \leq l \leq u. \tag{1}$$

But $TM = E^{cu} \oplus E^s$, where $E^{cu} := E^c \oplus E^u$, is also a dominated splitting for f^{-1}, so applying again Proposition 2 we have

$$\chi(\mathcal{T}(E^{cu}), f^{-1}) \leq h(f^{-1}) = h(f).$$

Again E^s is uniformly expanding for f^{-1}, so by Proposition 3 we have

$$\log(sp(f^{-1}_{*,s})) \leq \chi(\mathcal{T}(E^{cu}), f^{-1}), \quad \forall\, 0 \leq k \leq s.$$

Again, combining the two previous inequalities we get

$$\log(sp(f^{-1}_{*,k})) \leq h(f), \quad \forall\, 0 \leq k \leq s. \tag{2}$$

But now we assumed that M is orientable, and by duality we get

$$sp(f_{*,m-k}) = sp(f^{-1}_{*,k}),$$

which together with relation (2) implies that

$$\log(sp(f_{*,l})) \leq h(f), \quad \forall\, u+1 \leq l \leq m.$$

Combining this with relation (1) we get

$$\log(sp(f_{*,l})) \leq h(f), \quad \forall\, 0 \leq l \leq m,$$

or

$$\log(sp(f_*)) \leq h(f). \qquad \square$$

We remark that we didn't use any conditions about the integrability of the center, center-stable or center unstable distributions. Also we obtained actually strict inequalities for dimensions different from u and $u+1$, i. e.

$$\log(sp(f_{*,l})) < h(f), \quad \forall\, 0 \leq l \leq m,\ l \neq u, u+1.$$

This proofs can be applied to any partially hyperbolic diffeomorphism to give that

$$\log(sp(f_{*,l})) \leq h(f), \quad \forall\, l \in \{0, 1, \ldots, u-1, u, m-s, m-s+1, \ldots, m-1, m\}.$$

If the dimension of the center distribution is c then we get the desired inequalities for all the dimensions with the exception of $c-1$ of them: the dimensions $u+1, u+2, \ldots, u+c-1 = m-s-1$.

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